Lecture 3: Irreducibility over Finite Fields and Frobenius Automorphisms

Goal: Learn how to determine irreducibility of modular representations over finite fields, understand the Frobenius automorphism and how it gives rise to Galois conjugate representations, and examine the field of definition of characters.

1. Irreducibility over Finite Fields

Problem: Given a representation $\rho : G \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, how can we tell whether it is irreducible?

Definition 3.1 (Absolutely Irreducible Representation). A representation $\rho : G \to \mathrm{GL}_n(F)$ is *absolutely irreducible* if it remains irreducible over any finite extension of F.

Lemma 3.2. A representation ρ is absolutely irreducible over $\mathbb{F}_q$ if and only if its character cannot be written as a sum of the characters of two nontrivial subrepresentations over any extension of $\mathbb{F}_q$.

Theorem 3.3 (GCD Irreducibility Criterion). Let $\chi \in \mathrm{Irr}(G)$ be an ordinary irreducible character of a finite group G, and let p be a prime. If:

$$\gcd\left(\frac{|G|}{\chi(1)}, p\right) = 1,$$

then the reduction modulo p of the representation affording χ is irreducible over $\overline{\mathbb{F}}_p$.

Sketch of Proof: Use that the representation can be made integral and apply the semisimplicity of modules when $p \nmid |G|/\chi(1)$.

2. Field of Definition

Definition 3.4. The *field of definition* of a representation ρ over $\overline{\mathbb{F}}_p$ is the smallest finite field $\mathbb{F}_q \subset \overline{\mathbb{F}}_p$ such that ρ is equivalent to a representation defined over $\mathbb{F}_q$.

Example 3.5. If ρ has trace values $\{\alpha, \alpha^2\}$ with $\alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2$, then the field of definition is $\mathbb{F}_4$.

Lemma 3.6. The field generated by the character values (traces) of ρ contains the field of definition.

3. Frobenius Automorphism and Galois Conjugacy

Definition 3.7 (Frobenius Automorphism). Let $\mathbb{F}_q$ be a finite field. The Frobenius automorphism is the map:

$$\mathrm{Frob}_q : x \mapsto x^q.$$

Theorem 3.8. Let $\rho : G \to \mathrm{GL}_n(\mathbb{F}_{q^r})$ be an irreducible representation. Then the Frobenius twist:

$$\rho^{(q)}(g) := \rho(g)^q$$

defines another representation, which is Galois conjugate to ρ. These are not isomorphic unless ρ is defined over $\mathbb{F}_q$.

Example 3.9. Let ρ have character values in $\mathbb{F}_4$, and let $p = 2$. Then $\rho^{(2)}$ is another representation with conjugate trace values under $x \mapsto x^2$.

4. Frobenius Fixed Point Argument

Lemma 3.10. Let ρ be defined over $\mathbb{F}_q$ and let $\rho^{(q)} \cong \rho$. Then the representation is realizable over $\mathbb{F}_q$.

Application: If $\rho \ncong \rho^{(q)}$, then ρ and $\rho^{(q)}$ form a Galois pair. This occurs in modular representations when a representation over $\mathbb{F}_4$ is twisted to give a second irreducible representation.

5. Examples

Example 3.11 (Frobenius on $\mathbb{F}_4$). Let $\mathbb{F}_4 = \mathbb{F}_2(\omega)$ with $\omega^2 + \omega + 1 = 0$. Then:

$$\text{Frob}_2(\omega) = \omega^2$$

and:

$$\omega \mapsto \omega^2 \mapsto \omega^4 = \omega.$$

The Galois group is cyclic of order 2. Representations with traces in $\mathbb{F}_4 \setminus \mathbb{F}_2$ form Frobenius pairs.

6. Counterexamples

Counterexample 3.12. There exist representations over $\overline{\mathbb{F}}_p$ that are irreducible but not absolutely irreducible. For example, define a representation over $\mathbb{F}_p$ that splits over $\mathbb{F}_{p^2}$.

7. Summary

In this lecture, we've shown:

- How to use GCD criteria to detect irreducibility
- The role of character values in determining the field of definition
- How Frobenius twists create Galois-conjugate representations
- Why traces in $\mathbb{F}_4$ (as in A_5) lead to modular multiplicity and deeper structure

Coming Up in Lecture 4: We will introduce the *Brauer character table*, define *p-regular classes*, and explore the construction of *decomposition matrices* using character tables and modular data.